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# The Arcsine Law, Quantum-Classical Correspondence, Orthogonal Polynomials, And All That

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On the occasion of the 60th birthday of Professor Nobuaki Obata

## 1 What's the Arcsine Law?

The arcsine law is the probability distribution defined by

$$d\mu_{As}(x)=\frac{dx}{\pi\sqrt{2-x^{2}}}, \quad \left(-\sqrt{2}<x<\sqrt{2}\right).$$

It appears in the study of random walks/Brouwnian motions, Algebraic probability (Quantum probability, Noncommutative probability) such as monotone CLT and Quantum-Classical correspondence. It also play a crucial roles in Quantum Walks. On the other hand, it also appears in the context of orthogonal polynomials and number theory. In the present note we focus on the relationship between the Arcsine law, Quantum-Classical correspondence, Orthogonal Polynomials and all that(especially Quantum Walks).

The arcsine law can be characterization by moments :

$$E(X^{2m})=\frac{1}{2^{m}}\binom{2m}{m}, \quad E(X^{2m+1})=0$$

It is the solution for determinate moment problem because it has compact support. In such cases, moment convergence implies weak convergence.

## 2 Algebraic Probability

Algebraic probablity (Quantum probability, Noncommutative probablity) is a gereralization of probability theory in terms of algebraic probability space.

Algebraic probability space is a pair of an "algebra of quantities" and a "state" on that. Here "Algebra of quantities" means $*$-algebra i.e. algebra over complex numbers with "involution" $x \in \mathcal{A} \mapsto x^* \in \mathcal{A}$ such that for any $X, Y \in \mathcal{A}$ and $\alpha \in \mathbb{C}$

$$\begin{aligned}(X^*)^* = X, (\alpha X)^* = \overline{\alpha} X^*,\\ (X+Y)^* = X^* + Y^*,\\ (XY)^* = Y^* X^*.\end{aligned}$$

In short, the operation $*$ is a generalization of Hermite conjugate.

Let $\mathcal{A}$ be a (unital) $*$-algebra. A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying

$$\varphi(1) = 1, \quad \varphi(X^* X) \geq 0, \quad \text{for } X \in \mathcal{A}$$

is called a state on $\mathcal{A}$.

An algebraic probability space is a pair $(\mathcal{A}, \varphi)$ of $*$-algebra and state on thatElements in $\mathcal{A}$ are called the algebraic random variables in $(\mathcal{A}, \varphi)$. When $X = X^*$, it is said to be real.

We introduce a notation for the relationship among a state $\varphi : \mathcal{A} \to \mathbb{C}$, algebraic random variable $X \in \mathcal{A}$ and probability measure $\mu$ on $\mathbb{R}$:We denote $X \sim_\varphi \mu$ when $\varphi(X^m) = \displaystyle\int_{\mathbb{R}} x^m d\mu(x)$ for $m \in \mathbb{N}$.It is read that "under $\varphi$, $X$ obeys $\mu$." Such probability law exists for any real algebraic random variable. Uniqueness is up to moment problem.

# 3 Quantum-Classical Correspondence for Harmonic Oscillator

A quantum harmonic oscillator is a triple $(\Gamma(\mathbb{C}), a, a^*)$ such that

- $\Gamma(\mathbb{C}) := \oplus_{n=0}^{\infty} \mathbb{C}\Phi_n$ : A pre-Hilbert space defined by the inner product $< \Phi_n, \Phi_m >= \delta_{n,m}$

- anihillation operator $a$

$$a\Phi_0 = 0, \quad a\Phi_n = \sqrt{n}\Phi_{n-1} (n \geq 1)$$

- creation operator $a^*$

$$a^*\Phi_n = \sqrt{n+1}\Phi_{n+1}.$$

Let us consider the algebraic probability spaces $(\mathcal{A}, \varphi_n(\cdot))$ for quantum harmonic oscillator: Here, $\mathcal{A}$ denotes the $*$-algebra generated by $a, a^*$and $\varphi_n(\cdot)$: $\varphi(\cdot) :=< \Phi_n, (\cdot)\Phi_n >$ denotes tha state on it.

For $X := a + a^*$, it is well known that

$$X \sim_{\varphi_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

Here a question arises: When $n$ goes to infinity, what will happen?The answer is the theorem below.

**Theorem 3.1.** *Let $\mu_N$ be a probability distribution on $\mathbb{R}$ such that*

$$\frac{X}{\sqrt{2N+1}} \sim_{\varphi_N} \mu_N.$$

*Then $\mu_N$ weakly converges to $\mu_{As}$.*

This is nothing but Q-C correspondence for harmonic oscillator.

Saigo(2012) gave a simple proof from the viewpoint of algebraic probability, and it can be generalized!

# 4 Generalization to Interacting Fock Spaces

Let us introduce the notion of "interacting Fock space", which is a generalization of the quantum harmonic oscillator:

**Definition 4.1** (Jacobi sequence). A pair of sequences $(\{\omega_{n+1/2}\}, \{\alpha_n\})$ is called a Jacobi sequence,

- if $\{\omega_{n+1/2}\}$ are positive real numbers $0 < \omega_{1/2}$, $\omega_{3/2}$, $\omega_{5/2}$, $\cdots$ labeled by half natural numbers, and
- if $\{\alpha_n\}$ are real numbers $\alpha_0$, $\alpha_1$, $\alpha_2$, $\cdots$ labeled by natural numbers.

In other works as (Hora-Obata2007), the sequence $\{\omega_{n+1/2}\}$ is called a Jacobi sequence of infinite type and given different labels.

**Definition 4.2** (Interacting Fock space). Let $(\{\omega_{n+1/2}\}, \{\alpha_n\})$ be a Jacobi sequence. An *interacting Fock space* $\Gamma_{\omega,\alpha}$ is a complex pre-Hilbert space $\Gamma(\mathbb{C})$ equipped with the following additional structure $(\{\Phi_n\}_{n=0}^{\infty}, A, B, C)$:

- Fixed sequence of vectors $\{\Phi_n\}_{n=0}^{\infty} \subset \Gamma(\mathbb{C})$ satisfying

- $\langle \Phi_n, \Phi_m \rangle = 0$ if $m \neq n$, and $\langle \Phi_n, \Phi_n \rangle = 1$,
- $\Gamma(\mathbb{C})$ is a complex linear span of $\{\Phi_n\}$,

- $A, B, C\colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C})$ are linear operators uniquely determined by
  - $A\Phi_0 = 0$, $A\Phi_n = \sqrt{\omega_{n-1/2}}\,\Phi_{n-1}$.
  - $B\Phi_n = \alpha_n \Phi_n$.
  - $C\Phi_n = \sqrt{\omega_{n+1/2}}\,\Phi_{n+1}$.

The sequence of vectors $\{\Phi_n\}_{n=0}^{\infty} \subset \Gamma(\mathbb{C})$ forms a orthonormal set of $\Gamma(\mathbb{C})$. The operator $A$ is called the *annihilation* operator, $B$ is called the *preservation* operator, and $C$ is called the *creation* operator.

**Definition 4.3.** The summation $X = A+B+C$ is expressed by the following symmetric tridiagonal matrix:

$$X = \begin{pmatrix} \alpha_0 & \sqrt{\omega_{1/2}} & 0 & \cdots \\ \sqrt{\omega_{1/2}} & \alpha_1 & \sqrt{\omega_{3/2}} & \ddots \\ 0 & \sqrt{\omega_{3/2}} & \alpha_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

This is called the Jacobi matrix.

The sequence of real numbers $\langle X^m\Phi_0, \Phi_0 \rangle$ is called the moments sequence of the Jacobi matrix $X$. Accardi and Bożejko showed in (Accardi-Bożejko1998) that for every probability measure $\mu$ on $\mathbb{R}$ whose moments are finite, the moment sequence $M_m = \int_{\mathbb{R}} x^m d\mu(x)$ can be realized as that of an interacting Fock space $\langle X^m\Phi_0, \Phi_0 \rangle$.

Let $\mathcal{A}$ be the complex algebra generated by the matrices $A, B, C$ and by the identity matrix id. The multiplication and the linear structure are defined by the usual matrix calculations. The $*$-operation is given by the composition of transpose and complex conjugation. Since the generating set $\{A = C^*, B = B^*, C = A^*\} \subset \mathcal{A}$ is closed under the $*$-operation, the whole algebra $\mathcal{A}$ is also colsed under the $*$-operation.

Recall that the operators $A, B, C$ act on the linear space $\oplus_{n=0}^{\infty}\mathbb{C}\Phi_n$. Let $\varphi_k$ be the state defined as $\varphi_k(\cdot) := \langle \,\cdot\, \Phi_k, \Phi_k \rangle$. Then the pairs $\{(\mathcal{A}, \varphi_k)\}_{k\in\mathbb{N}}$ are algebraic probability spaces labeled by $k$.

Let $\Gamma_{\omega,\alpha} := (\Gamma(\mathbb{C}), A, B, C)$ be an interacting Fock space. For $X := A + B + C$, let us define $\mu_n$ by

$$\frac{X - \alpha_n}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}} \sim_{\varphi_n} \mu_n.$$

A probablity law $\mu$ is called the classical limit distribution of $\Gamma_{\omega,\alpha}$ if $\mu_n$ converge to $\mu$ in moments.

**Theorem 4.4.** *(Saigo-Sako2016)*
*Let $\Gamma_{\omega,\alpha} := (\Gamma(\mathbb{C}), A, B, C)$ be an interacting Fock space satisfying the condition* (RAC1) *below. Then the Arcsine law $\dfrac{dx}{\pi\sqrt{2-x^2}}$ is the classical limit distribution.*

Here (RAC1) means the condition that $A, B, C$ are "relatively asymptotically commutative". More precisely,

$$\lim_{n\to\infty} \frac{AC - CA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0,$$

$$\lim_{n\to\infty} \frac{AB - BA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0.$$

This condition is equivalent to :

$$\lim_{n\to\infty} \frac{\omega_{n+1/2}}{\omega_{n-1/2}} = 1, \quad \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}} = 0.$$

As an application, we can deduce an asymptotic behaviour of orthogonal polynomials.

# 5 Application

Let $\mu$ be a probability law on $\mathbb{R}$ with finite moments and $\{p_n(x)\}_{n=0,1,\cdots}$ be the $n$-th (monic) orthogonal polynomial for $\mu$. There exist the sequence $\alpha_n$, $\omega_{n-1/2}$ satisfying the so-called "three-term recurrence relations":

$$\begin{aligned} p_0(x) &= 1, \\ xp_0(x) &= p_1(x) + \alpha_0 p_0(x), \\ xp_n(x) &= p_{n+1}(x) + \alpha_n p_n(x) + \omega_{n-1/2}\, p_{n-1}(x), n \geq 1. \end{aligned}$$

If the support of $\mu$ is infinite, $\omega_{n-1/2}$ is always positive. In short, from $\mu$ we obtain "Jacobi sequence" $(\{\omega_{n+1/2}\}, \{\alpha_n\})$.Let $P_n$ be the $n$-th normalized polynomial, i.e. $p_n/\|p_n\|_2$. There exists isometry $U : \Gamma_{\omega,\alpha} \to L^2(\mathbb{R}, \mu)$ : $\Phi_n \mapsto P_n$ such that $U^* x U = A + B + C$. Here, $x$ is multiplication operator on $L^2(\mathbb{R}, \mu)$.

In short, a (measure theoretic) random variables can be decomposed into three noncommutative algebraic random variables. ("quantum decomposition".)

From $U^* x U = A + B + C$ we obtain $A + B + C \sim_{\varphi_n} |P_n(x)|^2 \mu(dx)$.

**Theorem 5.1.** *(Saigo-Sako2016) When the three-term recurrence relations satisfy (RAC1), $|P_n(x)|^2\mu(dx)$ weakly converge to the Arcsine law thorough the normalization (average 0, variance 1).*

Almost all "Famous" polynomials (Hermite, q-Hermite, Jacobi, Laguerre,.. and all that) satisfies (RAC1).

Then, what kinf of generalization is possible? (Partial) Answer : If we consider the condition (RAC2) below, we obtain the new kind of classical limit distribution, which is closely related to continuous time quantum walk/discrete Schrödinger equation. We call them "discrete Arcsine laws".

Here (RAC2) means that asymptotically $A, C$ are commutative and $[A, B]$ (and then $[C, B]$ also) become scalar.More precisely,

- $\lim_{n\to\infty} \frac{AC - CA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0$
- There exists a real number $r$ such that

$$\lim_{n\to\infty} \frac{(AB - BA) - rA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0.$$

(RAC2) can be represented in terms of Jacobi sequence $\{\omega, \alpha\}$ as below:

(RAC2) is equivalent to the condition below:

$\lim_{n\to\infty} \frac{\omega_{n+1/2}}{\omega_{n-1/2}} = 1$ and $\left\{\frac{\alpha_n - \alpha_{n-1}}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}}\right\}_n$ converge.We denote the limit of $\left\{\frac{\alpha_n - \alpha_{n-1}}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}}\right\}_n$ by $c$. (RAC1) is the $c = 0$ case in (RAC2).

**Theorem 5.2.** *(Saigo-Sako2016) For the cases $c \neq 0$ in (RAC2), $|P_n(x)|^2\mu(dx)$ converge to the discrete Arcsine law $\mu_c$ through the normalization (average 0, variance 1).The support of $\mu_c$ is $c\mathbb{Z}$ and for $n = 0, 1, 2, \cdots$ are*

$$\mu_c(\{cn\}) = \mu_c(\{-cn\}) = \left\{J_n\left(\frac{\sqrt{2}}{c}\right)\right\}^2.$$

*$J_n$: the n-th Bessel function of 1st kind.*

In fact, the discrete Arcsine law $\mu_c$ is equal to the distribution of continuous time quantum walk on $\mathbb{Z}$ at the time $t = \frac{1}{c}$ (Konno2005).When $c \to 0$, i.e. $t \to \infty$, what is the limit of $\mu_c$?

**Theorem 5.3.** *(Saigo-Sako2016) When $c \to 0$, $\mu_c$ weakly converge to the Arcsine law.*

The results above represents a fundamental relationship between the discrete Arcsine laws and the Arcsine law. At the same time, it relates different kinds of orthogonal polynomials. Moreover, it also gives an another proof of the central limit theorem for continuous time quantum walk/ discrete Schrödinger equation! To sum up, algebraic probability shed the new light on The Arcsine Law, Quantum-Classical Correspondence, Orthogonal Polynomials, And All That!

### References


- (Accardi-Bożejko1998): L. Accardi and M. Bożejko, Interacting Fock spaces and Gaussianization of probability measures, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1** (1998), 663-670.
- (Hora-Obata2007): A. Hora and N. Obata, *Quantum Probability and Spectral Analysis of Graphs*, Theoretical and Mathematical Physics, Springer, 2007.
- (Konno2005): N.Konno, Limit theorem for continuous-time quantum walk on the line, PRE(3) **72**, no.2.
- (Konno-Saigo-Sako2017): On a property of the simple random walk on $\mathbb{Z}$, available on arXiv:1701.07946
- (Saigo2012): H.Saigo, A new look at the arcsine law and "Quantum-Classical Correspondence", IDAQP **15**, no.3.
- (Saigo-Sako2016): H.Saigo and H.Sako, The arcsine law and an asymptotic behavior of orthogonal polynomials, AIHPD **4**, no.4.